Levy's Theorem

This is a key result and is used often during the course. The proof presented will use some results from probability theory:

Definition: Let X be a random variable. The characteristic function of X is its Fourier Transform;

$$\Psi_X(u) = \mathbb{E}(e^{iuX}) \quad , u \in \mathbb{R}, \; i^2 = -1 .$$

You may be unfamiliar with integrating complex valued functions. In fact it is quite straightforward,

$$e^{iuX(\omega)} = \cos(uX(\omega)) + i \sin(uX(\omega))$$

where $\cos(uX(\omega))$ is the real part and $\sin(uX(\omega))$ the imaginary part and both of these are real valued random variables. One can define

$$\mathbb{E}(e^{iuX}) = \mathbb{E}(\cos(uX)) + i\,\mathbb{E}(\sin(uX)) .$$

For random variables, X, Y, the joint characteristic function of X and Y is

$$\Psi_{X,Y}(u,v) = \mathbb{E}(e^{iuX} \cdot e^{ivY}) .$$

Theorem

X and Y are independent if and only if

$$\psi_{X,Y}(u,v) = \psi_X(u)\,\psi_Y(v)$$

Pf Not offered

Theorem

The random variable X is $N(\mu,\sigma^2)$ iff

$$\psi_X(u) = e^{i\mu u - \frac{1}{2}\sigma^2 u^2}$$

"Pf" You may be familiar with the result,

$$\mathbb{E}(h(X)) = \int_{-\infty}^{\infty} h(x) f_X(x)\, dx$$

For a "nice" function $h(x)$ and a continuous random variable X with density $f_X(x)$. We can extend this formula to "nice" complex valued functions and

$$h(x) = e^{iux}$$

is one of these.

So, $\mathbb{E}\left(e^{iuX}\right) = \int_{-\infty}^{\infty} e^{iux} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

You now exercise a familiar trick with normal distributions; the exponential term is

$$iux - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

You complete the square;

$$\frac{-1}{2\sigma^2}\left\{-2iux\sigma^2 + x^2 - 2x\mu + \mu^2\right\}$$

goes to

$$\frac{-1}{2\sigma^2}\left\{x^2 - 2(iu\sigma^2 + \mu)x + \mu^2\right\}$$

goes to

$$\frac{-1}{2\sigma^2}\left\{x^2 - 2(iu\sigma^2 + \mu) + (iu\sigma^2 + \mu)^2 - (iu\sigma^2 + \mu)^2 + \mu^2\right\}$$

goes to

$$-\frac{1}{2}\left(\frac{x-(iu\sigma^2+\mu)}{\sigma}\right)^2 + iu\mu - \frac{1}{2}u^2\sigma^2.$$

The first term you recognise as a normal density albeit with a complex mean. This "business part of the integral" gives 1 and leaves

$$e^{iu\mu - \frac{1}{2}u^2\sigma^2}$$

remaining. The converse follows from the inversion formula for Fourier Transforms. We refer to P. Billingsley; Probability and Measure. J. Wiley.
To precis the result; 'The Fourier Transform uniquely determines the distribution'. For a random variable, X, with distribution $F_X(x)$;

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} \, dF_X(x)$$

$$= \mathbb{E}(e^{itX}).$$

Remark: We already have Itô's Lemma for real valued functions. By applying this to the real and imaginary parts of complex functions of a real variable we can extend Itô's Lemma to such functions.

Theorem (P. Levy)

Let (X_t) be a continuous (local) martingale with $\langle X \rangle_t = t$. Then (X_t) is a Brownian Motion.

<u>Proof</u>

Let $f(t,x) = e^{iux + \frac{u^2 t}{2}}$.

By Itô's Lemma

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(s,X_s)\,ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)\,dX_s$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)\,d\langle X\rangle_s .$$

Now, $\frac{\partial f}{\partial t}(s,X_s) = \frac{1}{2}u^2 f(s,X_s)$,

$$\frac{\partial f}{\partial x}(s,X_s) = iu\, f(s,X_s),$$

$$\frac{\partial^2 f}{\partial x^2}(s,X_s) = i^2u^2 f(s,X_s) .$$

So,

$$f(t,X_t) = f(0,X_0) + \int_0^t \tfrac{1}{2}u^2 f(s,X_s)\,ds + \int_0^t iu\, f(s,X_s)\,dX_s$$
$$+ \frac{1}{2}\int_0^t i^2u^2 f(s,X_s)\,ds ,$$

as $\langle X\rangle_s = s$. As $i^2 = 1$, we get

$$f(t,X_t) = f(0,X_0) + iu\int_0^t f(s,X_s)\,dX_s .$$

Since (X_t) is a (local) martingale so is $f(t,X_t)$. Proceeding as if
(†) $f(t,X_t)$ is a martingale we have for $s < t$,

$$M_s\big(f(t,X_t)\big) = f(s,X_s)$$

that is,

$$M_s\left(e^{iuX_t + u^2\frac{t}{2}}\right) = e^{iuX_s + u^2\frac{s}{2}}$$

that is,

$$M_s\left(e^{iu(X_t - X_s)}\right) = e^{-u^2\frac{(t-s)}{2}}$$

(a number!, and $X_t - X_s$ is $N(0, t-s)$!)

We are going to show that $X_t - X_s$ is independent of $\mathcal{F}_s$.

Let $\Psi_{X_t - X_s, Y}(u, v) = \mathbb{E}\left(e^{iu(X_t - X_s)} e^{ivY}\right)$

$$= \mathbb{E}\left(M_s\left(e^{iu(X_t - X_s)} e^{ivY}\right)\right)$$

$= \mathbb{E}\left(M_s\left(e^{iu(X_t - X_s)}\right) e^{ivY}\right)$ when Y is $\mathcal{F}_s$ measurable

$= \mathbb{E}\left(e^{ivY}\right) e^{-\frac{u^2}{2}(t-s)}$ since $M_s\left(e^{iu(X_t - X_s)}\right)$ is a number

$= \mathbb{E}\left(e^{ivY}\right)\mathbb{E}\left(e^{iu(X_t - X_s)}\right)$ because $\mathbb{E}\left(M_s\left(e^{iu(X_t - X_s)}\right)\right) = \mathbb{E}\left(e^{iu(X_t - X_s)}\right)$,

$$= \Psi_Y(v)\,\Psi_{X_t - X_s}(u).$$

So $X_t - X_s$ is independent of $\mathcal{F}_s$
(take $Y = I_E$, $E \in \mathcal{F}_s$).

(f) Note that $f(s, X_s)$ is bounded and continuous — albeit complex valued — so that it satisfies

$$|f(s, X_s)| = e^{\frac{u^2 s}{2}}$$

and note the right side is independent of $\omega \in \Omega$. So $|f(s,X_s)|^2 = e^{u^2 s}$ and you can verify that,

$$\mathbb{E}\left(\int_0^t |f(s, X_s)|^2 ds\right) < \infty, \forall t.$$

The Multidimensional Version.

The steps for this result are:

For an $\mathbb{R}^d$ valued random variable, $\underline{X}$, define for $\underline{u} \in \mathbb{R}^d$,

$$\psi_{\underline{X}}(\underline{u}) = \mathbb{E}(e^{i\underline{u}\cdot\underline{X}})$$

where $u \cdot x$ denotes the 'dot' product of vectors. Let $\underline{X}$ and $\underline{Y}$ be $\mathbb{R}^d$ valued random variables, their joint characteristic functions is

$$\psi_{\underline{X},\underline{Y}}(\underline{u},\underline{v}) = \mathbb{E}(e^{i\underline{u}\cdot X}\, e^{i\underline{v}\cdot\underline{Y}}).$$

Theorem: $\underline{X}$ and $\underline{Y}$ are independent iff;

$$\psi_{\underline{X},\underline{Y}}(\underline{u},\underline{v}) = \psi_{\underline{X}}(\underline{u})\,\psi_{\underline{Y}}(\underline{v}) \quad .$$

Theorem: $\underline{X}$ is $N(\underline{\mu}, V)$ iff

$$\psi_{\underline{X}}(\underline{u}) = e^{i\underline{\mu}\cdot\underline{u} - \frac{1}{2}\underline{u}^{\tau} V \underline{u}} .$$

Remark: As we have seen, there is a multidimensional Itô rule. So now let $\underline{M} = (M_t^1, \ldots, M_t^d)$

be a continuous martingale with $\langle \underline{M} \rangle = t I_d$ (I_d is the $d \times d$ identity matrix).

One defines $f_{\underline{u}}(t, \underline{x}) = e^{i \underline{u} \cdot \underline{x} + \underline{u} \cdot \underline{u} \frac{t}{2}}$

then,

$$f_{\underline{u}}(t, \underline{M}_t) = f(0, \underline{M}_0) + \int_0^t \frac{\partial f}{\partial t}(s, \underline{M}_s)\, ds + \int_0^t \nabla_{\underline{x}} f(s, \underline{M}_s) \cdot d\underline{M}_s + \frac{1}{2} \int_0^t D^2 f \circ d[\underline{M}]_s$$

where $[\underline{M}] = [\langle M^i, M^j \rangle]$ and we know $\langle M^i, M^j \rangle = 0$ when $i \neq j$ and it t when $i = j$. So things simplify;

$$\frac{\partial f}{\partial t} = \frac{1}{2} |\underline{u}|^2 f, \quad \frac{\partial f}{\partial x_i} = i u_i f,$$

$$\frac{\partial^2 f}{\partial x_i^2} = i^2 u_i^2 f.$$

Substituting and adding up gives that $f(t, \underline{M}_t)$ is a martingale. Now proceed just as before.